Critical exponents of a two-dimensional continuum percolation system

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A simulation is carried out in order to investigate percolation threshold and critical exponents of a two-dimensional continuum percolation system. In the simulation, many circular holes are made in a rectangular plane at random until all of the paths from one end to the other disappear. The critical exponent $\nu=1.64$ for the correlation length is obtained on the basis of a scaling assumption, and the behavior of connectivity and percolation probability near the percolation threshold for an infinite sample is estimated by applying the finite-size scaling. [S1063-651X(96)02210-6]

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I. INTRODUCTION

The percolation problem [1–5] has been studied from various points of view, from theoretical interest to wide applications. For instance, it is well known that the percolation problem is an important model which indicates critical phenomena and retains various applicabilities which spread over various fields such as disease propagation [1], oil recovery in oil wells [6], random network [7,8], and so on.

Generally, discrete percolation models on a lattice have been discussed by many researchers, and the critical exponents of discrete models are clarified. However, only a few of the continuum models, where the positions of constituents are not restricted to the discrete sites of a regular lattice, have been investigated. Halperin, Feng, and Sen [9] predicted that some of the critical exponents in the continuum percolation model might be different from those in the conventional discrete percolation models. Therefore, the continuum percolation problem should be studied in detail.

The present report discusses the percolation threshold, the critical exponents of connectivity, and those of the percolation probability of a two-dimensional continuum percolation system by a simulation method. In the present simulation, many circular holes are made in a rectangular plane at random until all of the paths from one end to the other disappear. Variations of the connectivity and the percolation probability with the area of remaining parts of a plane are obtained for various values of diameter of holes, and the percolation threshold is estimated. The critical exponents of the connectivity and the percolation probability in the two-dimensional continuum model are obtained on the basis of the scaling assumption [10]. In particular the value of the critical exponent ν for the correlation length of the two-dimensional continuum percolation problem is 1.64, and this value differs from $\nu=4/3$ which is known for the lattice percolation [11]. Using the finite-size scaling [10], it is also shown how the connectivity and the percolation probability behave near the percolation threshold for an infinite sample.

II. NUMERICAL SIMULATION

Figure 1 shows a two-dimensional continuum percolation system. In this system, many circular holes are made in a rectangular plane at random. a and b indicate the length of edges of the rectangular plane, and d the diameter of a hole. ‘‘Remaining connected component’’ and ‘‘island’’ are remaining parts of the plane. As the number of holes increases, the area of remaining connected components which keep the connection between the edges OA and BC decreases, and the number of small islands increases.

In the simulation, the number of holes is increased at random until all of the paths from OA to BC are disconnected, and then areas of remaining parts of the plane r_a and the area of the largest remaining connected component r_c at the critical situation where the path between OA and BC disappears are measured. The ratio of r_a to the area of the rectangular plane, $p=r_a/ab$, means the probability that an arbitrary point in the plane belongs to any remaining parts of the plane. On the other hand, the ratio of r_c to ab indicates the probability that an arbitrary point in the plane belongs to the largest remaining connected component, and represents the approximate value of the percolation probability P.

Disconnections of paths are determined by investigating whether the overlaps between the holes percolates across the sample as follows. The maximum value max(y_j) and the minimum value min(y_j) are found out from coordinates of centers of holes (x_j,y_j) which unite with each other. If the conditions given by

$$\min(y_j)\leqslant\frac{d}{2}\quad\text{and}\quad\max(y_j)\geqslant\left(a-\frac{d}{2}\right)\tag{1}$$

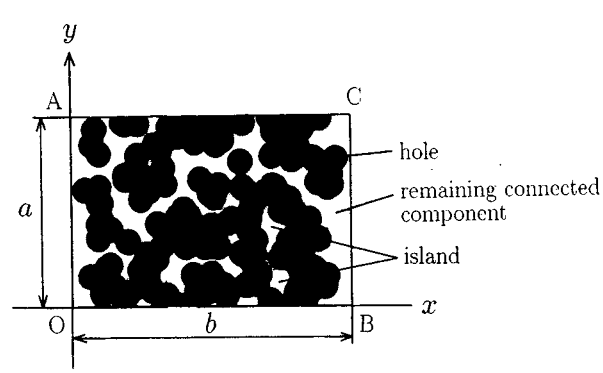

FIG. 1. Two-dimensional continuum percolation system.

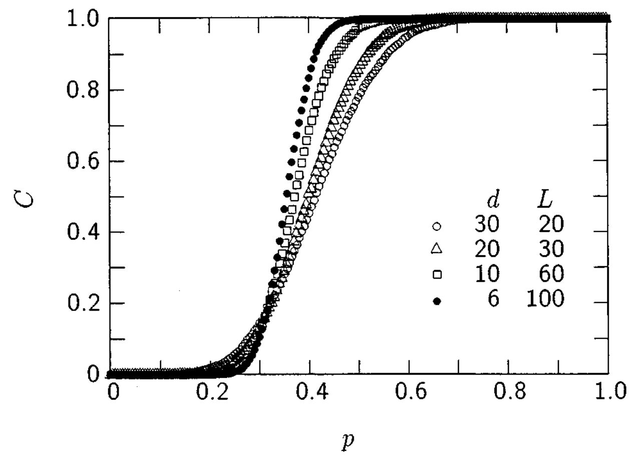

FIG. 2. The variation of C with p.

are satisfied, the path between OA and BC disappears. Areas of remaining parts are measured in the following manner. The plane $OABC$ is divided into 150×150 pieces, and the state of each piece is expressed by the value of a element of a matrix of order (150,150), 0 or 1. Initial values of all elements of the matrix are 1, and values of elements which correspond to holes are replaced by 0. Afterward clusters are labeled by using the cluster labeling method [12], and approximate values of areas of clusters are obtained from the number of elements whose values are equal to 1. In this case calculated values of areas are slightly smaller than the correct values. However the error for $p=0.1$ is about 3%, and the error decreases with the increase of p.

Furthermore, we obtain the connectivity C, which represents the probability of the existence of any remaining connected components which percolate through the system. The connectivity C in the interval $(p_i, p_i+\Delta p)$ is given by

$$C=f_i/n, \tag{2}$$

where n is the total number of samples whose area ratio p lies in the interval $(p_i, p_i+\Delta p)$, and f_i is the number of samples in which there are any path between OA and BC.

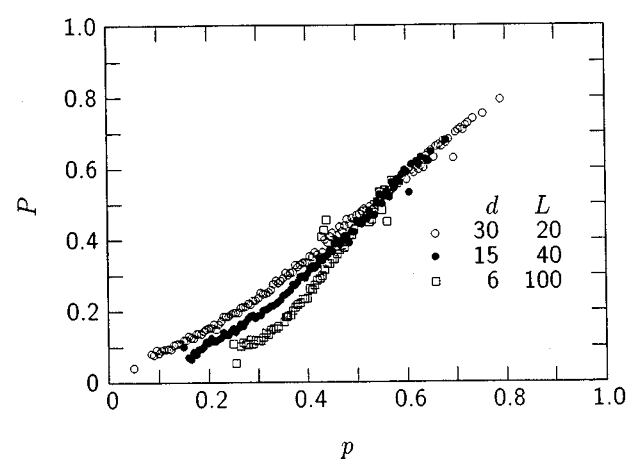

FIG. 3. The variation of P with p.

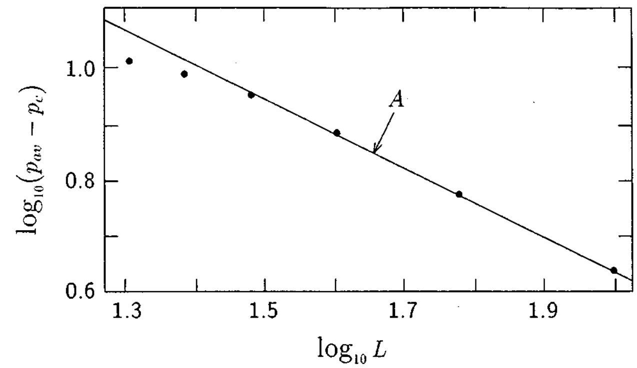

FIG. 4. The variation of $\log_{10}(p_{av}-p_c)$ with $\log_{10}L$ for the connectivity and the percolation probability.

III. NUMERICAL RESULTS

We divided the interval (0,1) of p into 200 equal small segments p_{si} $(i=1\sim 200)$, and carried out the numerical simulation by using the following values: length of OA, $a=100$ mm; length of OB, $b=100$ mm; diameter of holes, $d=6,10,15,20,25,30$ mm; total number of samples, $n=5000$. The interval of the segment p_{si} is $(p_i, p_i+0.005)$, where $p_i=(i-1)/200$.

A. Connectivity

Figure 2 shows the variation of the connectivity C with the area ratio p for different diameters d. Values of the relative system size L, which denotes the relative length of an edge of a plane when the value of d is fixed, are represented in Fig. 2. In this case, we set the value of L for $d=6$ equal to 100 for the sake of convenience. The increase of d means the decrease of L. As shown in this figure, the slope of curves increases with decreasing d, and simultaneously the threshold value for each diameter of holes, which is the minimum value of p for $C(p)\neq 0$, increases with the decrease of d. Furthermore, it can be seen that all lines intersect in the neighborhood of $p=0.315$. Thus we can estimate that the threshold value p_c for the infinite value of L may be 0.315.

B. Percolation probability

Figure 3 shows the variation of the percolation probability P with the ratio of the area of all remaining parts of a plane

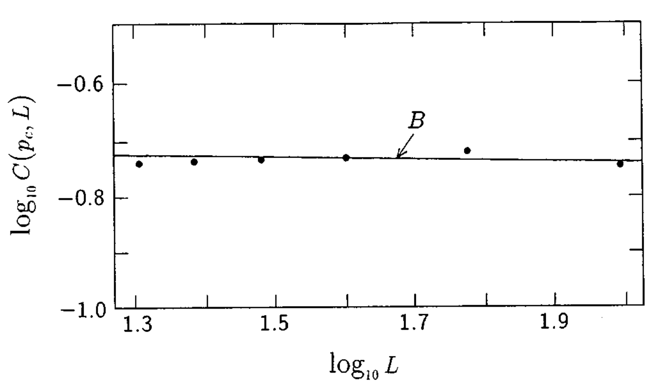

FIG. 5. The variation of $\log_{10}C(p_c, L)$ with $\log_{10}L$ for the connectivity.

TABLE I. Effective percolation threshold p_{av}.

d	L	p_{av}
6	100	0.36
10	60	0.37
15	40	0.39
20	30	0.40
25	24	0.41
30	20	0.42

p. Values of P are average over each small segment p_{si} for $d=6$, 15, and 30 mm which are indicated by □, ●, and ○, respectively.

For larger values of p, P is found to be proportional to p, and the area of the largest remaining connected component which connects between the edges OA and BC becomes large. Threshold values $p_c(L)$ increase with the increase of L. Referring to the result for the connectivity, the threshold value p_c for the infinite value of L is 0.315.

IV. CRITICAL EXPONENTS AND FINITE-SIZE SCALING

In this section, we obtained the critical exponents α, β, γ, and ν on the basis of the scaling assumption, and estimated the behavior of the connectivity and the percolation probability for $L\to\infty$ from applying the finite-size scaling. α, β, γ, and ν are the exponents which are corresponding to the total number of remaining connected components, the strength of the infinite network, the average size of finite remaining connected components, and the correlation length, respectively.

A. Critical exponents

According to the scaling assumption, the difference between the effective percolation threshold for one relative system size p_{av} and the threshold value p_c is given by

$$p_{av}-p_c \propto L^{-1/\nu}. \tag{3}$$

From Eq. (3), the critical exponent ν can be determined by the dependence of p_{av} upon L.

In the present case, the value of p_c is 0.315, and p_{av} can be calculated from

$$p_{av}=\sum_{i=1}^{200} p_i f_i/n. \tag{4}$$

Values of p_{av} are shown in Table I. We plot $\log_{10}(p_{av}-p_c)$ versus $\log_{10}L$, and draw the best straight line A for large $L(L=30,40,60,100)$ by the use of the method of least squares in Fig. 4. The value of the slope of the line A is -0.609. Thus we can obtain the critical exponent ν as follows:

$$\nu=1.64. \tag{5}$$

The value of β/ν can be also determined on the basis of the following dependence of $C(p_c,L)$ and $P(p_c,L)$ upon L.

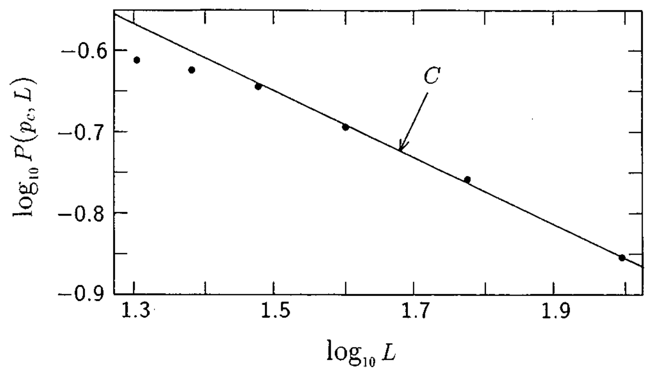

FIG. 6. The variation of $\log_{10}P(p_c,L)$ with $\log_{10}L$ for the percolation probability.

$$C(p_c,L)\propto L^{-\beta_c/\nu}, \quad P(p_c,L)\propto L^{-\beta_p/\nu}, \tag{6}$$

where the subscripts c and p indicate the values of the connectivity and the percolation probability, respectively. Since the qualitative behavior of $C(p)$ near $p=p_c$ is similar to those of the probability $P(p)$ and the magnetization $m(p)$ [13], we applied the symbol β to the exponent for $C(p)$. Log-log plots of $C(p_c,L)$ vs L and $P(p_c,L)$ vs L are shown in Figs. 5 and 6, respectively. The straight lines B and C show the best fit for large L $(L=30,40,60,100)$ obtained by the method of least squares. We can obtain values of β_c/ν and β_p/ν by the slope of the lines B and C. Using Eq. (5), values of the critical exponents β_c and β_p can be obtained as follows:

$$\beta_c=0.026, \quad \beta_p=0.67. \tag{7}$$

We obtain the exponents α and γ by the use of the scaling law

$$\alpha=2-D\nu, \quad \gamma=D\nu-2\beta, \tag{8}$$

where D is the dimension. Substituting Eqs. (5), (7) and $D=2$ into Eqs. (8), values of α, γ_c, and γ_p can be obtained.

$$\alpha=-1.28, \quad \gamma_c=3.23, \quad \gamma_p=1.94. \tag{9}$$

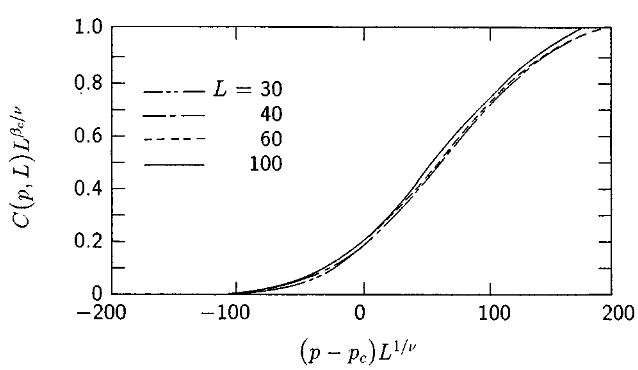

FIG. 7. The finite-size scaling of the connectivity. $\beta_c=0.026$, $\nu=1.64$, $p_c=0.315$.

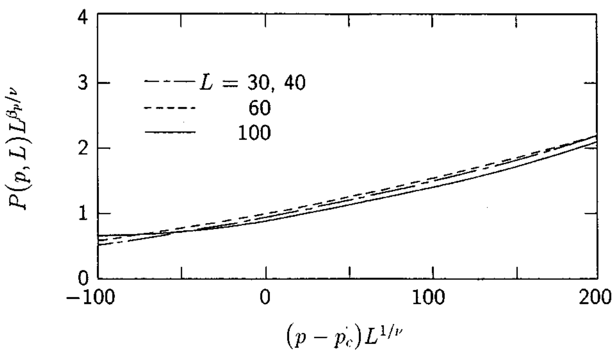

FIG. 8. The finite-size scaling of the percolation probability. $\beta_p = 0.67$, $\nu = 1.64$, $p_c = 0.315$.

As shown in Eqs. (5), (7), and (9), values of β and γ of the connectivity are different from those of the percolation probability.

It is known that the value of the critical exponent ν of a two-dimensional lattice model is 4/3 [11]. However, the value of the critical exponent ν of a continuum percolation system is 1.64, and this value is different from the value of a discrete model.

B. Finite-size scaling

Figures 7 and 8 show the results of the connectivity and the percolation probability which are arranged on the basis of the scaling assumption. The scaling of the percolation probability is carried out by using the curves of the fourth order polynomial which are determined by the least mean square approximation. As shown in these figures, all lines approach one another near $p = p_c$, and the relation of the finite-size scaling holds well. Therefore, it can be considered that the foregoing values of the critical exponents are appropriate, and we can estimate the behavior of the connectivity and the percolation probability near the percolation threshold for an infinite sample from Figs. 7 and 8.

V. CONCLUSION

The threshold value and the critical exponents of the two-dimensional continuum percolation system have been investigated by the use of computer simulation. The threshold value of the connectivity and the percolation probability has been estimated, and the critical exponents have been obtained on the basis of the scaling assumption. Furthermore, the finite-size scaling has been carried out.

The following conclusions can be drawn from the obtained results.

(1) The threshold value of the connectivity and the percolation probability is 0.315 in the present continuum percolation model. (2) The critical exponents of the connectivity and the percolation probability are as follows:

connectivity;

$$\alpha = -1.28, \quad \beta_c = 0.026, \quad \gamma_c = 3.23, \quad \nu = 1.64,$$

percolation probability;

$$\alpha = -1.28, \quad \beta_p = 0.67, \quad \gamma_p = 1.94, \quad \nu = 1.64.$$

(3) The relation of the finite-size scaling holds well near the percolation threshold, and the behavior of the connectivity and the percolation probability near the percolation threshold for an infinite sample can be estimated.

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[1] S.R. Broadbent and J.M. Hammersley, Proc. Cambridge Philos. Soc. **53**, 629 (1957).

[2] V.K.S. Shante and S. Kirkpatrick, Adv. Phys. **20**, 325 (1971).

[3] D. Stauffer, *Introduction to Percolation Theory* (Taylor & Francis, London, 1985).

[4] T. Odagaki, *Introduction to Percolation Physics* (Shokabo, Tokyo, 1993).

[5] S. Miyazima, K. Maruyama, and K. Okumura, J. Phys. Soc. Jpn. **60**, 2805 (1991).

[6] D. Wilkinson and J.F. Willmsen, J. Phys. A **16**, 3365 (1983).

[7] S. Kirkpatrick, Rev. Mod. Phys. **45**, 574 (1973).

[8] B.P. Watson and P.L. Leath, Phys. Rev. B **9**, 4893 (1974).

[9] B.I. Halperin, S. Feng, and P.N. Sen, Phys. Rev. Lett. **54**, 2391 (1985).

[10] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, 2nd ed. (Taylor & Francis, London, 1991), p. 70.

[11] A. Bunde and S. Havlin, *Fractals and Disordered Systems* (Springer-Verlag, Berlin, 1991), p. 52.

[12] J. Hoshen and R. Kopelman, Phys. Rev. B **14**, 3438 (1976).

[13] A. Bunde and S. Havlin, *Fractals and Disordered Systems* (Springer-Verlag, Berlin, 1991), p. 55.